Lecture 5 (1/12/22).

- Hadamard's 3 circles Thm from Lecture 4 notes.

Phragmén-Lindelöf Principle.

The "principle" is a way abstracting the method of proof of Thm 1 from last Lecture.

Thm 1. Let G be simply connected and suppose $\exists$ analytic φ in G s.t. $\varphi \neq 0$ and $|\varphi| \leq N$. Assume f is analytic in G, $\partial_\infty G = A \cup B$, and

- $\limsup_{z \to \xi} |f(z)| \leq M$, $\xi \in A$
- $\limsup_{z \to \xi} |f(z)| |\varphi(z)|^\eta \leq M$, $\xi \in B$, for all $\eta > 0$.

Then $|f| \leq M$ in G.

Rem. Idea is that $\varphi(z) \to 0$ as $z \to B$, allowing $|f| \to \infty$ in the assumption.

Pf. Idea is to use MMT-III to the analytic functions $g_\eta(z) = f(z)\varphi(z)^\eta$.

$e^{\eta \log \varphi(z)}$

$\log \varphi$ is OK since G is s.c. and $\varphi \neq 0$.

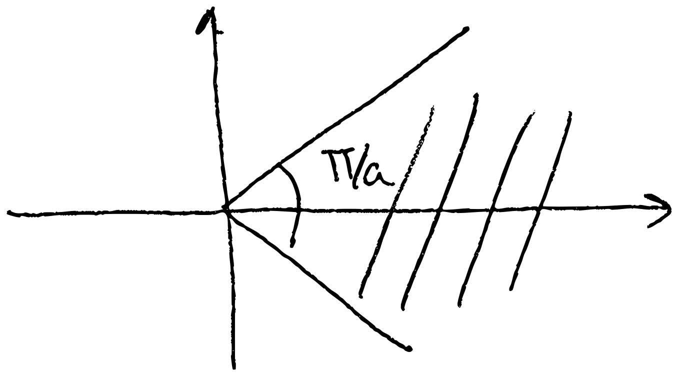
Note that $|g_\eta| \leq |f|\,|\varphi|^\eta \leq N^\eta |f|$. Thus, if $\xi \in A$, $\limsup_{z \to \xi} |g_\eta| \leq N^\eta M$. If $\xi \in B$, then $\limsup_{z \to \xi} |g_\eta(z)| \leq M$ by assumption. Thus, by MMT-III, $|g_\eta| \leq \max(M, N^\eta M)$. Fix $z_0 \in G \Rightarrow$

$$|f(z_0)| \leq \max(M, N^\eta M)\,|\varphi(z_0)|^{-\eta}$$

$\neq 0$ by assumption.

This estimate holds for all $0 < \eta < \eta_0$. Letting $\eta \to 0$, $N^\eta \to 1$, $|\varphi(z_0)|^{-\eta} \to 1$ $\Rightarrow$ $|f(z_0)| \leq M$. Since z_0 arbitrary in G $\Rightarrow$ $|f| \leq M$ as desired. ☒

Cor 1. Let $G = \{z = re^{i\theta} : |\theta| < \frac{\pi}{2a}\}$, $a \geq \frac{1}{2}$:



If f is analytic in G, $\limsup\limits_{z \to \xi \in \partial G} |f(z)| \leq M$

and, for some $b < a$,

$$|f(z)| \leq A e^{|z|^b},$$

then $|f| \leq M$ in G.

Pf. Use Thm 1, so need $\underbrace{\varphi(z)}_{}$ and $A \cup B = \partial_\infty G$. Take $b < c < a$, and let $\varphi(z) = e^{-z^c}$, $z^c = e^{c \log z}$. Let $A = \partial G$ and $B = \infty \in \partial_\infty G$.

The conclusion follows from Thm 1 if we can show

$$\limsup_{z \to \infty} |f(z)|\,|\varphi(z)|^{\eta} \le M.$$

Well,

$$|f(z)|\,|\varphi(z)|^{\eta} \le A e^{|z|^b} \cdot e^{-\operatorname{Re} z^c \cdot \eta}.$$

If $z = re^{i\theta}$, then $\operatorname{Re} z^c = r^c \cos c\theta$. Since

$$|\theta| < \frac{\pi}{2a} \Rightarrow |c\theta| < \underbrace{\frac{c}{a}}_{<1} \frac{\pi}{2} \Rightarrow \cos c\theta \ge \delta > 0.$$

$$\Rightarrow |f(z)|\,|\varphi(z)|^{\eta} \le A e^{|z|^b - \eta\delta|z|^c} \to 0.$$

Hence, Thm 1 $\Rightarrow$ $|f| \le M$ as desired.

<u>Rem</u> A little more trickery shows that

$$|f(z)| \le A e^{|z|^b}, \quad b < a$$

can be relaxed to: $\forall \delta > 0\ \exists A_\delta$ s.t.

$$|f(z)| \le A_\delta e^{\delta |z|^a}.$$

(See Conway).